Spherical structures on torus knots and links *

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Abstract

The present paper considers two infinite families of cone-manifolds endowed with spherical metric. The singular strata is either the torus knot $\mathrm{t}(2n+1,2)$ or the torus link $\mathrm{t}(2n,2)$. Domains of existence for a spherical metric are found in terms of cone angles and volume formulæ are presented.

Key words: Spherical geometry, cone-manifold, knot, link.

1 Introduction

A three-dimensional cone-manifold is a metric space obtained from a collection of disjoint simplices in the space of constant sectional curvature k by isometric identification of their faces in such a combinatorial fashion that the resulting topological space is a manifold (also called the underlying space for a given cone-manifold).

Such the metric space inherits the metric of sectional curvature k on the union of its 2- and 3-dimensional cells. In case $k = +1$ the corresponding cone-manifold is called spherical (or admits a spherical structure). By analogy, one defines euclidean ($k = 0$) and hyperbolic ($k = -1$) cone-manifolds.

The metric structure around each 1-cell is determined by a cone angle that is the sum of dihedral angles of corresponding simplices sharing the 1-cell under identification. The singular locus of a cone-manifold is the closure of all its 1-cells with cone angle different from 2π. For the further account we suppose that every component of the singular locus is an embedded circle with constant cone angle along it.

A particular case of cone-manifold is an orbifold with cone angles $2\pi/m$, where m is an integer (cf. [1]).

The present paper considers two infinite families of cone-manifolds with underlying space the three-dimensional sphere $\mathbb{S}^3$. The first family consists of cone-manifolds with singular locus the torus knot $\mathrm{t}(2n+1,2)$ with $n \geq 1$. In the rational census [2] these knots are denoted by $(2n+1)/1$. The second family of cone-manifolds consists of those with singular locus a two-component torus link $\mathrm{t}(2n,2)$ with $n \geq 2$. These links are two-bridge and correspond to the links

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$2n/1$ in the rational census. The simplest examples of such the knots and links are the trefoil knot $3/1$ and the link $4/1$. In the Rolfsen table [2] one finds them as the knot 3_1 and the link 4_1^2.
By the Theorem of W. Thurston [3], the manifold $\mathbb{S}^3 \backslash 3_1$ does not admit a hyperbolic structure. However, it admits two other geometric structures [4]: $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$. It follows from the paper [5] that the spherical dodecahedron space (i.e. Poincaré homology sphere) is a cyclic 5-fold covering of $\mathbb{S}^3$ branched over 3_1. Thus, the orbifold $3_1(\frac{2\pi}{5})$ with singular locus the trefoil knot and cone angle $\frac{2\pi}{5}$ is spherical. Due to the Dunbar's census [6], orbifold $3_1(\frac{2\pi}{n})$ is spherical if $n \leq 5$, Nil-orbifold if $n = 6$ and $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$-orbifold if $n \geq 7$. Spherical structure on the cone-manifold $3_1(\alpha)$ with underlying space the three-dimensional sphere $\mathbb{S}^3$ is studied in [7].
The consideration of two-bridge torus links is carried out starting with the simplest one possessing non-abelian fundamental group, namely 4_1^2.
The previous investigation on spherical structures for cone-manifolds is carried out mainly in the papers [8, 9, 10]. The present paper develops a method to analyse existence of a spherical metric for two-bridge torus knot and link cone-manifolds. Also, the lengths of singular geodesics are calculated and the volume formulæ are obtained (cf. Theorem 1 and Theorem 2).

2 Projective model $\mathbb{S}^3_\lambda$

The purpose of the present section is to construct the projective model $\mathbb{S}^3_\lambda$ that one can use to study geometric properties of two-bridge torus knots and links and to build up holonomy representation for the corresponding cone-manifolds. Other projective models for homogeneous geometries are described in [11].
Consider the set $\mathbb{C}^2 = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}\}$ as a four-dimensional vector space over $\mathbb{R}$. We denote it by $\mathbb{C}^2_{\mathbb{R}}$ and equip with Hermitian product

$$\langle (z_1, z_2), (w_1, w_2) \rangle_{\mathrm{H}} = (z_1, z_2) \mathcal{H} \, \overline{(w_1, w_2)}^T,$$

where

$$\mathcal{H} = \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix}$$

is a symmetric matrix with $-1 < \lambda < +1$.
The natural inner product is associated to the Hermitian form above:

$$\langle (z_1, z_2), (w_1, w_2) \rangle = \mathrm{Re} \, \langle (z_1, z_2), (w_1, w_2) \rangle_{\mathrm{H}}$$

and the respective norm is

$$\|(z_1, z_2)\| = |z_1|^2 + |z_2|^2 + \lambda(z_1 \overline{z}_2 + \overline{z}_1 z_2).$$

Call two elements (z_1, z_2) and (w_1, w_2) in $\overset{\circ}{\mathbb{C}}{}^2_{\mathbb{R}} = \mathbb{C}^2_{\mathbb{R}} \setminus (0, 0)$ equivalent if there is $\mu > 0$ such that $(z_1, z_2) = (\mu \, w_1, \mu \, w_2)$. We denote this equivalence relation as $(z_1, z_2) \sim (w_1, w_2)$.

Identify the factor-space $\overset{\circ}{\mathbb{C}}{}^2_{\mathbb{R}}/\sim$ with the three-dimensional sphere

$$\mathbb{S}^3_\lambda = \{(z_1,\, z_2) \in \mathbb{C}^2_{\mathbb{R}} : \|(z_1,\, z_2)\| = 1\},$$

endowed with the Riemannian metric

$$\mathrm{d}s^2_\lambda = |\mathrm{dz}_1|^2 + |\mathrm{dz}_2|^2 + \lambda(\mathrm{dz}_1\mathrm{d}\overline{\mathrm{z}}_2 + \mathrm{d}\overline{\mathrm{z}}_1\mathrm{dz}_2).$$

By means of equality

$$\mathrm{d}s^2_\lambda = \frac{1+\lambda}{2}\,|\mathrm{d}z_1 + \mathrm{d}z_2|^2 + \frac{1-\lambda}{2}\,|\mathrm{d}z_1 - \mathrm{d}z_2|^2,$$

the linear transformation

$$\xi_1 = \sqrt{\frac{1+\lambda}{2}}\,(z_1 + z_2), \;\; \xi_2 = \sqrt{\frac{1-\lambda}{2}}\,(z_1 - z_2)$$

provides an isometry between $(\mathbb{S}^3_\lambda,\, \mathrm{d}s^2_\lambda)$ and $(\mathbb{S}^3,\, \mathrm{d}s^2)$, where $\mathrm{d}s^2 = |\mathrm{d}\xi_1|^2 + |\mathrm{d}\xi_2|^2$ is the standard metric of sectional curvature $+1$ on the unit sphere $\mathbb{S}^3 = \{(\xi_1,\, \xi_2) \in \mathbb{C}^2 : |\xi_1|^2 + |\xi_2|^2 = 1\}$.
Let P, Q be two points in $\mathbb{S}^3_\lambda$. The spherical distance between P and Q is a real number $d_\lambda(P,Q)$ that is uniquely determined by the conditions $0 \le d_\lambda(P,Q) \le \pi$ and $\cos d_\lambda(P,Q) = \langle P, Q\rangle$.

3 Torus knots $\mathbb{T}_n$

Let $\mathbb{T}_n, n \ge 1$ be the torus knot $\mathrm{t}(2n+1, 2)$ embedded in $\mathbb{S}^3$. The knot $\mathbb{T}_n$ is the two-bridge knot $(2n+1)/1$ in the rational census (Fig. 1). Let $\mathbb{T}_n(\alpha)$ denote a cone-manifold with singular locus $\mathbb{T}_n$ and the cone angle α along it.

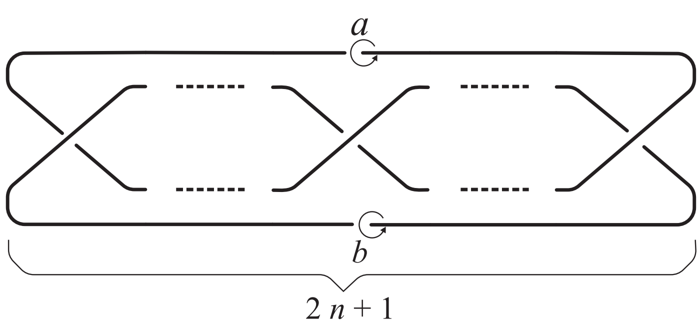

Figure 1: Knot $(2n+1)/1$

The aim of the present section is to investigate cone-manifolds $\mathbb{T}_n(\alpha)$, $n \ge 1$ to find out the domain of sphericity in terms of the cone angle and to derive the volume formulæ.
Two lemmas precede the further exposition:

Lemma 1 *For every* $0 < \alpha < 2\pi$ *and* $-1 < \lambda < +1$ *the linear transformations*

$$A = \begin{pmatrix} 1 & 0 \\ -2\,i\,e^{i\frac{\alpha}{2}}\lambda\,\sin\frac{\alpha}{2} & e^{i\alpha} \end{pmatrix}$$

and

$$B = \begin{pmatrix} e^{i\alpha} & -2\,i\,e^{i\frac{\alpha}{2}}\lambda\,\sin\frac{\alpha}{2} \\ 0 & 1 \end{pmatrix}$$

are isometries of $\mathbb{S}^3_\lambda$.

Proof. For the further account let us assume that the multiplication of vectors by matrices is to the right. A linear transformation L of the space $\mathbb{C}^2_\mathbb{R}$ preserves the corresponding Hermitian form if and only if for every pair of vectors $P, Q \in \mathbb{C}^2_\mathbb{R}$ it holds that

$$\langle P,\, Q\rangle_{\mathrm{H}} = P\mathcal{H}\overline{Q}^T = PL\mathcal{H}\overline{L}^T\overline{Q}^T = \langle PL,\, QL\rangle_{\mathrm{H}}.$$

The condition above is equivalent to

$$\mathcal{H} = L\mathcal{H}\overline{L}^T.$$

In particular,

$$\cos d_\lambda(P,Q) = \langle P,\, Q\rangle = \langle PL,\, QL\rangle = \cos d_\lambda(PL,QL),$$

that means L preserves the spherical distance between P and Q.
Let $L = A$ and $L = B$ in series, one verifies that A and B preserve the Hermitian norm on $\mathbb{C}^2_\mathbb{R}$ and, consequently, the spherical distance on $\mathbb{S}^3_\lambda$. $\square$

Lemma 2 *Let* A *and* B *be the same matrices as in the affirmation of Lemma 1. Then for all integer* $n \geq 1$ *one has*

$$(AB)^nA - B(AB)^n \;=\; 2\,U_{2n}(\Lambda)\,e^{i\,\frac{(2n+1)(\pi+\alpha)}{2}}\,\sin\frac{\alpha}{2}\,M,$$

where M *is a non-zero* 2×2*-matrix and* $U_{2n}(\Lambda)$ *is the second kind Chebyshev polynomial of power* $2n$ *in variable* $\Lambda \,=\, \lambda\,\sin\frac{\alpha}{2}$.

Proof. As far as $-1 < \lambda < +1$, one obtains

$$-1 \;<\; \Lambda \;=\; \lambda\,\sin\frac{\alpha}{2} \;<\; +1.$$

Substitute

$$\Lambda \;=\; \cos\theta,$$

with the unique $0 \;<\; \theta \;<\; \pi$.
Then matrices A and B are rewritten in the form

$$A = \begin{pmatrix} 1 & 0 \\ -2\,i\,e^{i\frac{\alpha}{2}}\cos\theta & e^{i\alpha} \end{pmatrix},$$

$$B = \begin{pmatrix} e^{i\alpha} & -2\,i\,e^{i\frac{\alpha}{2}}\cos\theta \\ 0 & 1 \end{pmatrix}.$$

On purpose to diagonalize the matrix AB, use

$$V = \begin{pmatrix} i\,e^{-i\frac{\alpha}{2}}\,e^{-i\theta} & i\,e^{-i\frac{\alpha}{2}}\,e^{i\theta} \\ 1 & 1 \end{pmatrix},$$

and obtain

$$D \,=\, V^{-1}(AB)V \,=\, \begin{pmatrix} -e^{i\alpha}\,e^{2i\theta} & 0 \\ 0 & -e^{i\alpha}\,e^{-2i\theta} \end{pmatrix}.$$

Note, that V might be not an isometry, but it is utile for computation.
Thus

$$\begin{aligned}(AB)^n A - B(AB)^n \,&=\, (V\,D^n\,V^{-1})A - B(V\,D^n\,V^{-1}) \,= \\ &=\, 2\,\frac{\sin(2n+1)\theta}{\sin\theta}\;e^{i\,\frac{(2n+1)(\pi+\alpha)}{2}}\;\sin\frac{\alpha}{2}\;\begin{pmatrix} -1 & \lambda \\ -\lambda & 1 \end{pmatrix} = \\ =\, 2\,U_{2n}(\cos\theta)\,e^{i\,\frac{(2n+1)(\pi+\alpha)}{2}}\;\sin\frac{\alpha}{2}\;M \,&=\, 2\,U_{2n}(\Lambda)\,e^{i\,\frac{(2n+1)(\pi+\alpha)}{2}}\;\sin\frac{\alpha}{2}\;M,\end{aligned}$$

with the matrix

$$M = \begin{pmatrix} -1 & \lambda \\ -\lambda & 1 \end{pmatrix}$$

as the present Lemma claims. □
The main theorem of the section follows:

Theorem 1 *The cone-manifold $\mathbb{T}_n(\alpha)$, $n \geq 1$ is spherical if*

$$\frac{2n-1}{2n+1}\,\pi < \alpha < 2\pi - \frac{2n-1}{2n+1}\,\pi.$$

The length of its singular geodesic (i.e. the length of the knot $\mathbb{T}_n$) equals

$$l_\alpha \,=\, (2n+1)\,\alpha \,-\, (2n-1)\,\pi.$$

The volume of $\mathbb{T}_n(\alpha)$ is

$$\mathbb{V}\mathrm{ol}\,\mathbb{T}_n(\alpha) \,=\, \frac{1}{2n+1}\left(\frac{2n+1}{2}\,\alpha \,-\, \frac{2n-1}{2}\,\pi\right)^2.$$

Proof. The fundamental group of the knot $\mathbb{T}_n$ is presented as

$$\pi_1(\mathbb{S}^3\backslash\mathbb{T}_n) = \langle a, b | (ab)^n a \,=\, b(ab)^n\rangle,$$

with generators a and b as at Fig. 1.
Since the cone-manifold $\mathbb{T}_n(\alpha)$ admits a spherical structure, then there exists a holonomy mapping [1], that is a homomorphism

$$h : \pi_1(\mathbb{S}^3\backslash\mathbb{T}_n) \longmapsto \mathrm{Isom}\;\mathbb{S}^3_\lambda.$$

We will choose h in respect with geometric construction of the cone-manifold. All the further computations to find the length of the knot $\mathbb{T}_n$ and the volume of the cone-manifold $\mathbb{T}_n(\alpha)$ are performed making use of the corresponding fundamental polyhedron $\mathcal{P}_n$ (Fig. 2). The construction algorithm for the polyhedron is given in [12].
The combinatorial polyhedron $\mathcal{P}_n$ has vertices P_i, $i \in \{1, \ldots, 4n+2\}$ and edges P_iP_{i+1}, $i \in \{1, \ldots, 4n+2\}$, with $P_{4n+3} = P_1$, also P_1P_{2n+2} and P_2P_{2n+3}. Let N, S denote the middle points (the North and the South poles of $\mathcal{P}_n$) on the edges P_1P_{2n+2} and P_2P_{2n+3}, respectively. Then, consider also edges NP_i, SP_i, $i \in \{1, \ldots, 4n+2\}$.
Without loss in generality, choose the holonomy representation such that

$$h(a) = A, \ h(b) = B,$$

where A and B are matrices from Lemma 1.
The generators of the fundamental group for $\mathbb{T}_n$ under the holonomy mapping h correspond to isometries acting on $\mathcal{P}_n$. These isometries identify its faces by means of rotation about the edge P_1P_{2n+2} for the top "cupola" of $\mathcal{P}_n$ and rotation about P_2P_{2n+3} for the bottom one (see, Fig. 2). Then the edges P_1P_{2n+2} and P_2P_{2n+3} knot itself to produce $\mathbb{T}_n$ (cf. [12, 13]).
In order to construct the polyhedron $\mathcal{P}_n$ assume that its edge P_1P_2 is given by

$$P_1 = (1, 0), \ P_2 = (0, 1).$$

Then one has

$$\cos d_\lambda(P_1, P_2) = \langle P_1, P_2 \rangle = \lambda,$$

i.e. the spherical distance between the points P_1 and P_2 can vary from 0 to π. Thus, prescribing certain coordinates to the end-points of the edge P_1P_2 we do not loss in generality of the consideration.
Note, that the axis of the isometry A from Lemma 1 contains P_1 and the axis of B contains P_2. The aim of the construction for the polyhedron $\mathcal{P}_n$ is to bring its edges P_1P_{2n+2} and P_2P_{2n+3} to be axes of the respective isometries A and B. The other vertices P_i has to be images of P_1 and P_2 under action of A and B. The polyhedron $\mathcal{P}_n$ is said to be proper if

(a) inner dihedral angles along P_1P_{2n+2} and P_2P_{2n+3} are equal to α;

(b) the following curvilinear faces are identified by A and B:

$$A : NP_1P_2 \ldots P_{2n+2} \to NP_1P_{4n+2} \ldots P_{2n+3}P_{2n+2},$$

$$B : SP_2P_1P_{4n+2} \ldots P_{2n+3} \to SP_2P_3 \ldots P_{2n+3};$$

(c) sum of the inner dihedral angles ψ_i along P_iP_{i+1}, $i \in \{1, \ldots, 4n+1\}$ equals 2π;

(d) sum of the dihedral angles ϕ_i for corresponding tetrahedra NSP_iP_{i+1}, $i \in \{1, \ldots, 4n+1\}$ at their common edge NS is 2π;

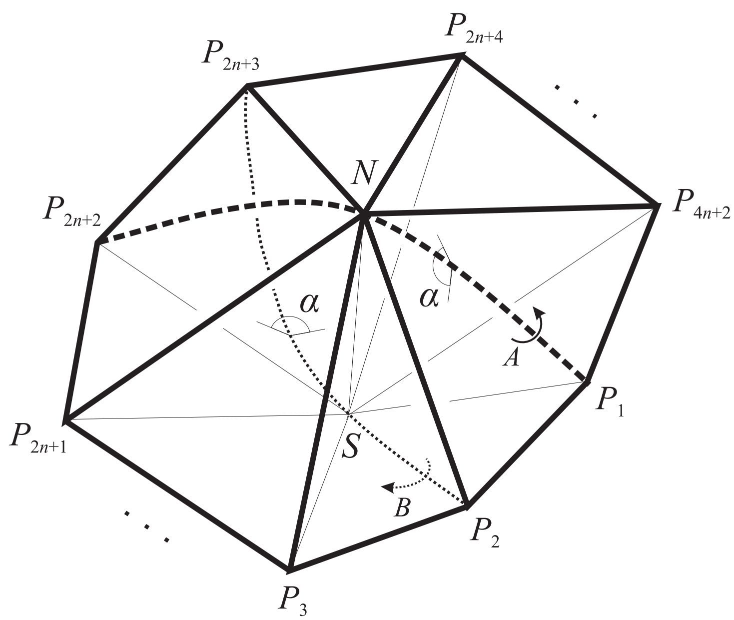

Figure 2: Fundamental polyhedron $\mathcal{P}_n$ for $\mathbb{T}_n(\alpha)$

(e) all the tetrahedra NSP_iP_{i+1} with $i \in \{1, \ldots, 4n+2\}$, $P_{4n+3} = P_1$ are non-degenerated and coherently oriented.

By the orientation of a tetrahedron NSP_iP_{i+1} one means the sign of the Gram determinant $\det(S, N, P_i, P_{i+1})$ for corresponding quadruple S, N, P_i, $P_{i+1} \in \mathbb{C}^2_{\mathbb{R}}$, where $i \in \{1, \ldots, 4n+2\}$, $P_{4n+3} = P_1$. A tetrahedron is non-degenerated if $\det(S, N, P_i, P_{i+1}) \neq 0$. Thus, claim (e) is satisfied if all the Gram determinants are non-zero and of the same sign.
If $\alpha = \frac{2\pi}{m}$, $m \in \mathbb{N}$, then due to the Poincaré Theorem [14, Theorem 13.5.3] claims (a) – (e) imply that the group generated by the isometries A and B is discreet and its presentation is

$$\Gamma = \langle A, B | (AB)^n A = B(AB)^n, A^m = B^m = \mathrm{id} \rangle.$$

The metric space $\mathbb{S}^3_\lambda / \Gamma \cong \mathbb{T}_n(\frac{2\pi}{m})$ is a spherical orbifold, and $\mathcal{P}_n$ is its fundamental polyhedron. If $m \notin \mathbb{N}$ then the group generated by A and B might be non-discreet. However, the identification for the faces of $\mathcal{P}_n$ is of the same fashion as if it were $m \in \mathbb{N}$ and as the result one obtains the cone-manifold $\mathbb{T}_n(\alpha)$.
By means of Lemma 1 and construction of $\mathcal{P}_n$ claims (a) and (b) are satisfied.
For the holonomy mapping h to exist the following relation should be satisfied:

$$h((ab)^n a) - h(b(ab)^n) \;=\; (AB)^n A - B(AB)^n \;=\; 0.$$

By Lemma 2, the condition above is satisfied if and only if

$$U_{2n}(\Lambda) = 0,$$

where $\Lambda = \lambda \sin \frac{\alpha}{2}$.
Thus, the parameter λ of the metric ds^2_λ is determined completely by a root of the polynomial $U_{2n}(\Lambda)$. From the above formula, λ is related to the cone angle α by means of the equality

$$\lambda = \frac{\Lambda}{\sin \frac{\alpha}{2}} .$$

The roots of $U_{2n}(\Lambda)$ are given by the following formula:

$$\Lambda_k = \cos \frac{k\pi}{2n+1} ,$$

with $k \in \{1, \ldots, 2n\}$.
The parameter λ for the metric ds^2_λ has to be chosen in order the polyhedron $\mathcal{P}_n$ be proper and the metric itself be spherical.
Note, that the edges P_iP_{i+1}, $i \in \{1, \ldots, 4n+2\}$, $P_{4n+3} = P_1$ are equivalent under action of the group $\Gamma = \langle A, B\rangle$. Thus, the relation $(AB)^n A = B(AB)^n$ implies the equality

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2k\pi,$$

where k is an integer.
Show that one can choose λ for the equality $k = 1$ to hold for all α in the affirmation of the Theorem. Due to the paper [15], every two-bridge knot cone-manifold with cone angle π is a spherical orbifold. In this case all the vertices P_i of the fundamental polyhedron belong to the same circle and all the dihedral angles ψ_i and ϕ_i are equal to each other [12]:

$$\phi_i = \psi_i = \frac{\pi}{2n+1} .$$

As far as $\cos d_\lambda(N, S) = \cos d_\lambda(P_i, P_{i+1}) = \lambda$, then in case $\alpha = \pi$ one obtains

$$\lambda = \frac{\Lambda_k}{\sin \frac{\pi}{2}} = \cos \theta$$

for certain $k \in \{1, \ldots, 2n\}$ and then

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2(2n+1)\theta.$$

Using the formula for the roots of $U_{2n}(\Lambda)$ obtain that

$$\sum_{i=1}^{2(2n+1)} \psi_i = 2k\pi$$

if $\alpha = \pi$. Thus, claim (c) for the polyhedron $\mathcal{P}_n$ with $\alpha = \pi$ is satisfied if $k = 1$. As far as the parameter α varies continuously and sum of the angles ψ_i

represents a multiple of 2π, one has that

$$\sum_{i=1}^{2(2n+1)} \psi_i \;=\; 2\pi$$

for all α.
By analogy, show that with

$$\lambda = \frac{\Lambda_1}{\sin\frac{\alpha}{2}}$$

the equality

$$\sum_{i=1}^{2(2n+1)} \phi_i \;=\; 2\pi$$

holds, that means claim (d) is also satisfied.
Verify that under conditions of the Theorem the metric $\mathrm{d}s^2_\lambda$ is spherical. This claim is equivalent to the inequality

$$-1 < \lambda < +1.$$

Note, that for

$$\frac{2n-1}{2n+1}\,\pi < \alpha < 2\pi - \frac{2n-1}{2n+1}\,\pi$$

it follows

$$\sin\frac{\alpha}{2} \;>\; \sin\frac{(2n-1)\pi}{2(2n+1)}\,.$$

As far as $\sin\frac{\alpha}{2} > 0$ and $\Lambda_1 \;=\; \sin\frac{(2n-1)\pi}{2(2n+1)} > 0$, one has

$$0 < \lambda < 1.$$

By analogy with Lemma 1 verify that

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is an isometry of $\mathrm{d}s^2_\lambda$.
Fixed point sets of A and B in $\mathbb{S}^3_\lambda$ are circles

$$\operatorname{Fix} A = \{(z_1, 0) : z_1 \in \mathbb{C}, |z_1| = 1\}$$

and

$$\operatorname{Fix} B = \{(0, z_2) : z_2 \in \mathbb{C}, |z_2| = 1\},$$

correspondingly. The geometric meaning of C is that it maps the first fixed circle to the other. Thus, the relation $B = CAC^{-1}$ holds.
The following equalities

$$P_{2k+1} \;=\; P_1(AB)^k,\; k \in \{0, \ldots, n\},$$

$$P_{2k} \;=\; P_2(AB)^{k-1},\; k\in\{1,\ldots,n+1\};$$

and

$$P_{2k+1} \;=\; P_1(BA)^{2n-k+1},\; k\in\{n+1,\ldots,2n\},$$
$$P_{2k} \;=\; P_2(BA)^{2n-k+2},\; k\in\{n+2,\ldots,2n+1\},$$

follow from the identification scheme of the edges of $\mathcal{P}_n$.
Define the auxiliary function

$$\varepsilon(m) = \frac{m}{2}\,\alpha - \frac{4n-m}{2}\,\pi.$$

By analogy with the proof of Lemma 2 it follows that

$$(AB)^k \;=\; C(BA)^k C^{-1} \;=$$

$$= \begin{pmatrix} -\frac{\sin(2k-1)\theta}{\sin\theta}\, e^{i\,\varepsilon(2k)} & -\frac{\sin 2k\theta}{\sin\theta}\, e^{i\,\varepsilon(2k-1)} \\ \frac{\sin 2k\theta}{\sin\theta}\, e^{i\,\varepsilon(2k+1)} & \frac{\sin(2k+1)\theta}{\sin\theta}\, e^{i\,\varepsilon(2k)} \end{pmatrix},$$

where $\theta = \frac{\pi}{2n+1}$.
Suppose N and S to be middle-points of the edges P_1P_{2n+2} and P_2P_{2n+3}, respectively. Then

$$N = (e^{i\,\frac{\varepsilon(2n+1)}{2}},\, 0),\;\; S = (0,\, e^{i\,\frac{\varepsilon(2n+1)}{2}}).$$

For the lengths l_α of the singular geodesic one has

$$\cos\frac{l_\alpha}{4} = \langle P_1, N\rangle = \langle P_1C, NC\rangle = \langle P_2, S\rangle.$$

Thus
$$\cos\frac{l_\alpha}{4} \;=\; \cos\frac{(2n+1)\alpha-(2n-1)\pi}{4}\,.$$

By construction of the polyhedron $\mathcal{P}_n$, the inequality $0 < l_\alpha < 4\pi$ holds. Then it follows
$$l_\alpha = (2n+1)\alpha - (2n-1)\pi.$$

Given the coordinates of the vertices P_i and the poles N and S of the polyhedron $\mathcal{P}_n$, verify claim (e).
For every four points $A, B, C, D \in \mathbb{C}^2_{\mathbb{R}}$, where

$$A = (A_1, A_2),\;\; B = (B_1, B_2),\;\; C = (C_1, C_2),\;\; D = (D_1, D_2),$$

their Gram determinant is

$$\det(A,B,C,D) := \det\begin{pmatrix} \operatorname{Re}A_1 & \operatorname{Im}A_1 & \operatorname{Re}A_2 & \operatorname{Im}A_2 \\ \operatorname{Re}B_1 & \operatorname{Im}B_1 & \operatorname{Re}B_2 & \operatorname{Im}B_2 \\ \operatorname{Re}C_1 & \operatorname{Im}C_1 & \operatorname{Re}C_2 & \operatorname{Im}C_2 \\ \operatorname{Re}D_1 & \operatorname{Im}D_1 & \operatorname{Re}D_2 & \operatorname{Im}D_2 \end{pmatrix}.$$

Each tetrahedron NSP_iP_{i+1} with $i \in \{1,\ldots,2n+1\}$ is isometric to $NSP_{2n+i+1}P_{2n+i+2}$, $i \in \{1,\ldots,2n+1\}$, $P_{4n+3} = P_1$ by means of the isometry C defined above. Thus, we consider only the tetrahedra NSP_iP_{i+1} with $i \in \{1,\ldots,2n+1\}$. Split them into two groups: the tetrahedra $NSP_{2k+1}P_{2k+2}$ with $k \in \{0,\ldots,n\}$ and the tetrahedra $NSP_{2k}P_{2k+1}$ with $k \in \{1,\ldots,n\}$.
Substitute $\alpha = \beta + \pi$ and proceed with straightforward calculations:

$$\Delta_k^{(1)}(\beta) = \det(S, N, P_{2k+1}, P_{2k+2}) = \cos^2 \frac{L_1\,\beta}{4} - U_{2k-1}^2(\cos\theta)\,\sin^2\frac{\beta}{2} =$$

$$= T_{L_1}^2(\cos\frac{\beta}{4}) - U_{2k-1}^2(\cos\theta)\,\sin^2\frac{\beta}{2},$$

where $k \in \{0,\ldots,n\}$, $L_1 = |2n-4k+1|$, $\theta = \frac{\pi}{2n+1}$, $\beta \in [-2\,\theta, 2\,\theta]$;

$$\Delta_k^{(2)}(\beta) = \det(S, N, P_{2k}, P_{2k+1}) = \cos^2 \frac{L_2\,\beta}{4} - U_{2k-2}^2(\cos\theta)\,\sin^2\frac{\beta}{2} =$$

$$= T_{L_2}^2(\cos\frac{\beta}{4}) - U_{2k-1}^2(\cos\theta)\,\sin^2\frac{\beta}{2},$$

where $k \in \{1,\ldots,n\}$, $L_2 = |2n-4k+3|$, θ and β the same as above. The first kind Chebyshev polynomial of degree $k \geq 0$ is denoted by T_k. Assume that

$$U_{-1}(\cos\theta) = 0, \;\; U_0(\cos\theta) = 1$$

for the sake of brevity.
All the functions $\Delta_k^{(j)}(\beta)$, $j \in \{1,2\}$ are even on the interval $[-2\theta, 2\theta]$. Then one considers them only on the interval $[0, 2\theta]$. Note, that the polynomial $T_{L_j}^2(\cos\beta)$ monotonously decreases and the function $\sin^2\frac{\beta}{2}$ monotonously increases with $\beta \in [0, 2\theta]$. Moreover, $T_{L_j}^2(\cos 0) = T_{L_j}^2(1) = 1$. Then it follows that $\Delta_k^{(j)}(\beta) > 0$ with $\beta \in (-2\theta, 2\theta)$. Also, one has $\Delta_k^{(j)}(\pm 2\,\theta) = 0$.
Then for all $\beta \in (-2\theta, 2\theta)$ (i.e. for all α in the affirmation of the Theorem)

$$\det(S, N, P_i, P_{i+1}) > 0$$

where $i \in \{1,\ldots,4n+2\}$, $P_{4n+3} = P_1$. Thus, claim (e) for the polyhedron $\mathcal{P}_n$ is satisfied.
Use the Schläfli formula [16] to obtain the volume formula for $\mathbb{T}_n(\alpha)$. One has

$$\mathrm{d}\mathbb{V}\mathrm{ol}\,\mathbb{T}_n(\alpha) \;=\; \frac{l_\alpha}{2}\,\mathrm{d}\alpha \;=\; \frac{(2n+1)\alpha - (2n-1)\pi}{2}\,\mathrm{d}\alpha.$$

Note, that $\mathbb{V}\mathrm{ol}\,\mathbb{T}_n(\alpha) \to 0$ with $\alpha \to \frac{2n-1}{2n+1}\,\pi$. In this case $d_\lambda(P_i, P_{i+1}) \to 0$, where $i \in \{1,\ldots,4n+2\}$, $P_{4n+3} = P_1$ and the fundamental polyhedron collapses to a point. Thus

$$\mathbb{V}\mathrm{ol}\,\mathbb{T}_n(\alpha) \;=\; \frac{1}{2n+1}\left(\frac{2n+1}{2}\,\alpha \;-\; \frac{2n-1}{2}\,\pi\right)^2.$$

□

Remark 1 *The domain of the spherical metric existence in Theorem 1 was indicated before in [10, Proposition 2.1].*

4 Torus links $\mathbb{L}_n$

Let $\mathbb{L}_n, n \geq 2$ be a torus link $\mathrm{t}(2n, 2)$ with two components. The corresponding link in the rational census is $2n/1$ (Fig. 3). The fundamental group of $\mathbb{L}_n$ is presented as

$$\pi_1(\mathbb{S}^3 \backslash \mathbb{L}_n) = \langle a, b | (ab)^n \; = \; (ba)^n \rangle.$$

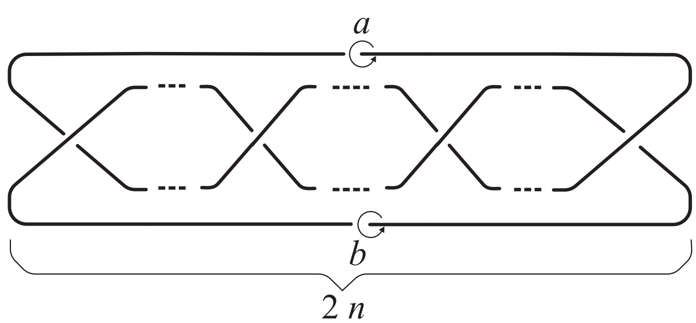

Figure 3: Link $2n/1$

Let $\mathbb{L}_n(\alpha,\, \beta)$ denote a cone-manifold with singular locus the link $\mathbb{L}_n$ and the cone angles $\alpha,\, \beta$ along its components.
For every $\alpha, \beta \in (0, 2\pi)$ and $\lambda \in (-1, +1)$, we denote

$$A = \begin{pmatrix} 1 & 0 \\ -2\, i\, e^{i\frac{\alpha}{2}} \lambda \, \sin\frac{\alpha}{2} & e^{i\alpha} \end{pmatrix}$$

and

$$B = \begin{pmatrix} e^{i\beta} & -2\, i\, e^{i\frac{\beta}{2}} \lambda \, \sin\frac{\beta}{2} \\ 0 & 1 \end{pmatrix}.$$

By Lemma 1, linear transformations A and B are isometries of $\mathbb{S}^3_\lambda$.

Lemma 3 *For every integer $n \geq 2$ the following equality holds*

$$(AB)^n - (BA)^n \; = \; 4\, U_{n-1}(\Lambda) \; \lambda \; e^{i(\frac{\alpha+\beta}{2}+\pi)\, n} \, \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \; M,$$

where M is a non-zero 2×2 matrix and $U_{n-1}(\Lambda)$ is the second kind Chebyshev polynomial of degree $n-1$ in variable

$$\Lambda \; = \; (1 - \lambda^2) \cos\frac{\alpha - \beta}{2} \; + \; \lambda^2 \cos\frac{\alpha + \beta}{2} \; .$$

Proof. By analogy with Lemma 2. □
With Lemma 3 the main theorem of the section follows:

Theorem 2 *The cone-manifold $\mathbb{L}_n(\alpha,\,\beta)$, $n \geq 2$ is spherical if*

$$-2\pi\left(1-\frac{1}{n}\right) \,<\, \alpha \,-\, \beta \,<\, 2\pi\left(1-\frac{1}{n}\right)\,,$$

$$2\pi\left(1-\frac{1}{n}\right) \,<\, \alpha \,+\, \beta \,<\, 2\pi\left(1+\frac{1}{n}\right)\,.$$

The lengths l_α, l_β of its singular geodesics (i.e. lengths of the components for $\mathbb{L}_n$) are equal to each other and

$$l_\alpha \,=\, l_\beta \,=\, \frac{\alpha+\beta}{2}\,n \,-\, \pi\,(n-1).$$

The volume of $\mathbb{L}_n(\alpha,\beta)$ is

$$\mathbb{V}\mathrm{ol}\,\mathbb{L}_n(\alpha,\,\beta) \,=\, \frac{1}{2n}\left(\frac{\alpha+\beta}{2}\,n \,-\, (n-1)\pi\right)^2.$$

Proof. One continues the proof by analogy with Theorem 1.
Suppose that $\mathbb{L}_n(\alpha,\beta)$ is spherical. Then there exists a holonomy mapping [1]:

$$h:\pi_1(\mathbb{S}^3\backslash\mathbb{L}_n)\longmapsto \mathrm{Isom}\;\mathbb{S}^3_\lambda,$$

$$h(a)=A,\;\; h(b)=B.$$

Also,

$$h((ab)^n)-h((ba)^n)=(AB)^n-(BA)^n=0.$$

By means of Lemma 3 the equality above holds either if $\lambda=0$, or if

$$\Lambda \,=\, (1-\lambda^2)\cos\frac{\alpha-\beta}{2} \,+\, \lambda^2\cos\frac{\alpha+\beta}{2}$$

is a root of the equation $U_{n-1}(\Lambda)=0$.
In case $\lambda=0$ the image of h is abelian, because of the additional relation $AB=BA$. With $n\geq 2$ this leads to a degenerate geometric structure. Thus, one has to choose the parameter λ for the metric $\mathrm{d}s^2_\lambda$ using roots of the Chebyshev polynomial $U_{n-1}(\Lambda)$.
The fundamental polyhedron $\mathcal{F}_n$ for the cone-manifold $\mathbb{L}_n(\alpha,\,\beta)$ is depicted at Fig. 4. Suppose its vertices P_1 and P_2 to be

$$P_1=(1,0),\; P_2=(0,1).$$

The axes of isometries A and B correspond to the edges P_1P_{2n+1} and P_2P_{2n+2}. Points N and S are respective middles of the edges P_1P_{2n+1} and P_2P_{2n+2}. Those are called North and South poles of the polyhedron.
The polyhedron $\mathcal{F}_n$ is said to be proper if

(a) respective inner dihedral angles along the edges P_1P_{2n+1} and P_2P_{2n+2} are equal to α and β;

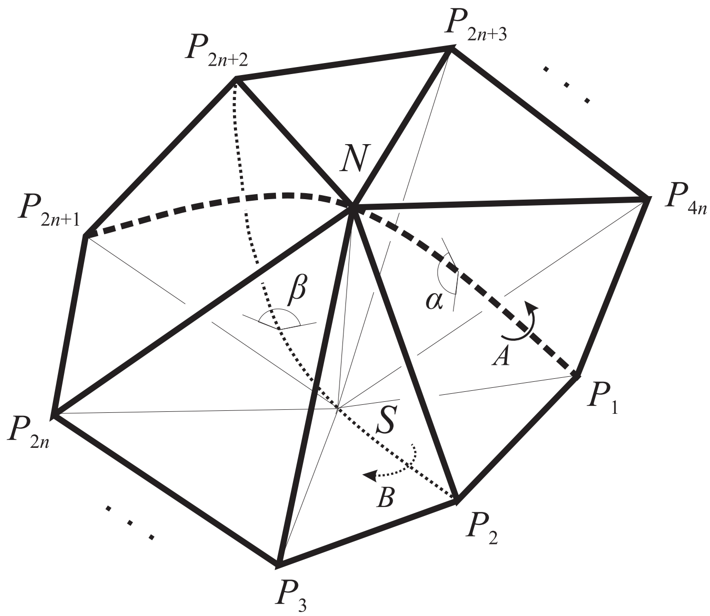

Figure 4: The fundamental polyhedron $\mathcal{F}_n$ for $\mathbb{L}_n(\alpha,\beta)$

(b) curvilinear faces of the polyhedron are identified by A and B:

$$A : NP_1P_2\dots P_{2n+1} \to NP_1P_{4n}\dots P_{2n+2}P_{2n+1},$$

$$B : SP_2P_1P_{4n}\dots P_{2n+2} \to SP_2P_3\dots P_{2n+2};$$

(c) sum of the inner dihedral angles ψ_i along the edges P_iP_{i+1}, $i \in \{1,\dots,4n-1\}$ equals 2π;

(d) sum of the dihedral angles ϕ_i for tetrahedra NSP_iP_{i+1}, $i \in \{1,\dots,4n-1\}$ at their common edge NS equals 2π;

(e) all the tetrahedra NSP_iP_{i+1} with $i \in \{1,\dots,4n\}$, $P_{4n+1} = P_1$ are non-degenerated and coherently oriented.

In order to choose the parameter λ for the corresponding metric consider the fundamental polyhedron $\mathcal{F}_n$ with $\alpha = \beta = \pi$. Then all its vertices belong to the same circle and all the dihedral angles ψ_i of the tetrahedra NSP_iP_{i+1} along the edges P_iP_{i+1} are equal to $\psi = \frac{\pi}{2n}$ [12]. Also the dihedral angles ϕ_i of the tetrahedra NSP_iP_{i+1} along their common edge NS are equal to each other:

$$\phi_i = \phi = \frac{\pi}{2n}.$$

In this case $\lambda = \langle P_1, P_2\rangle = \cos\phi$ and

$$\Lambda \;=\; -\cos 2\phi \;=\cos\frac{(n-1)\pi}{n}.$$

All the roots of $U_{n-1}(\Lambda)$ are given by the formula

$$\Lambda_k \ = \cos\frac{k\pi}{n}\,,\ k \in \{1,\ldots,n-1\},$$

so one choose the root Λ_k with $k = n-1$. Then, by analogy with Theorem 1, equalities

$$\sum_{i=1}^{4n} \psi_i \ = \ 2\pi$$

and

$$\sum_{i=1}^{4n} \phi_i \ = \ 2\pi$$

are satisfied at the point $\alpha = \beta = \pi$ of the domain

$$\mathcal{D} = \left\{(\alpha,\beta) : |\alpha-\beta| < 2\pi\left(1-\frac{1}{n}\right), |\alpha+\beta-2\pi| < \frac{2\pi}{n}\right\},$$

depicted at Fig. 5.
In terms of the parameter λ, that defines the metric $\mathrm{d}s^2_\lambda$, one has

$$\lambda^2 \ = \ \frac{\cos\frac{\alpha-\beta}{2} + \cos\frac{\pi}{n}}{\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}}\,.$$

As for all $(\alpha,\beta) \in \mathcal{D}$ the inequality $0 < \lambda^2 < 1$ is satisfied, the metric $\mathrm{d}s^2_\lambda$ is spherical regarding the corresponding domain. By analogy with Theorem 1 one can show that claims (a) – (d) for the polyhedron $\mathcal{F}_n$ are satisfied in the interior of $\mathcal{D}$.
The lengths l_α and l_β of singular geodesics for the cone-manifold $\mathbb{L}_n(\alpha,\beta)$ meet the relations

$$\cos\frac{l_\alpha}{2} = \langle P_1, N\rangle,$$

$$\cos\frac{l_\beta}{2} = \langle P_2, S\rangle.$$

By analogy with the proof of Theorem 1 one obtains

$$l_\alpha \ = \ l_\beta \ = \ \frac{\alpha+\beta}{2}\,n \ - \ \pi(n-1).$$

Given the coordinates of the vertices for the fundamental polyhedron verify claim (e) for all (α,β) in the domain $\mathcal{D}$.
Make use of the Schläfli formula [16] to obtain the volume of $\mathbb{L}_n(\alpha,\beta)$:

$$\mathrm{d}\,\mathbb{V}\mathrm{ol}\,\mathbb{L}_n(\alpha,\beta) \ = \ \frac{l_\alpha}{2}\,\mathrm{d}\alpha \ + \ \frac{l_\beta}{2}\,\mathrm{d}\beta \ = \ \left(\frac{\alpha+\beta}{2}\,n \ - \ \pi(n-1)\right)\,\mathrm{d}\left(\frac{\alpha+\beta}{2}\right).$$

Note, that with

$$\alpha = \beta \to \pi\,\frac{n-1}{n}$$

the fundamental polyhedron $\mathcal{F}_n$ collapses to a point (i.e. the volume tends to 0). The last affirmation of the Theorem follows. $\square$

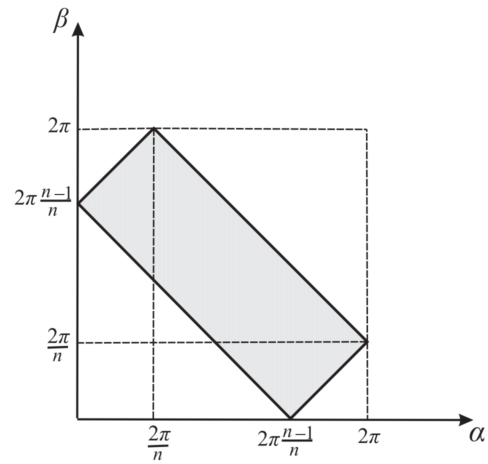

Figure 5: The domain $\mathcal{D}$ of sphericity for $\mathbb{L}_n(\alpha, \beta)$

Remark 2 *Under condition $\alpha = \beta$ the inequality from the affirmation of Theorem 2 coincides with the inequality from [10, Proposition 2.2].*

Remark 3 *Note, that the lengths of the singular geodesics for $\mathbb{L}_n(\alpha, \beta)$ are equal even if $\alpha \neq \beta$.*

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